A Study and Modelling of Liquid-Phase Mixing in a Flotation Column

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ABSTRACT


The mixing process of the liquid phase in a gas-liquid flotation column and the effect of the flowrates on it have been investigated. The increase of the gas and liquid flows had opposite effects: the former enhanced the mixing process, whereas the latter caused the behaviour of the column to approach plug flow.

A mixed zones-in-series model is proposed for interpreting the experimental results, which involves two parameters: the number of zones and the ratio of backwards to net liquid flow. In this way, the mixing was related to the backflow and the increase in mixing was due to an intense internal recirculation. The model was able to predict accurately the column behaviour in the experimental range investigated. Its major advantage lies in its structured description of the column contents.

INTRODUCTION

Bubble columns, commonly used as reactors and/or absorbers in the chemical industry, are finding lately new applications in the mineral processing and the effluent treatment plants as flotation columns. In the flotation process, as it is known, a liquid–solid slurry is fed into an aerated vessel; the solid particles collide with the bubbles and, if adhesion occurs, are eventually transported to the top of the vessel – given that no detachment occurs. There, a froth rich in these particles is generally formed and removed, whereas the carrying liquid and any remaining solids (e.g. gangue) are removed, usually from the bottom of the vessel.

Compared to the more widely used mechanically agitated cells, flotation columns may offer some advantages: the longer contact times achieved between gas and the solid–liquid slurry, the lack of mechanical agitation which may achieve savings in terms of energy required (Ek, 1986; Hyde and Stojsic, 1987).
and reduction of the maintenance costs, the countercurrent flow pattern offering better attachment probability, etc. The latter would also be advantageous, as compared to the concurrent mode of operation, due to a highly efficient "filtration" effect and the provision of a quiescent exit – which is most important in effluent treatment, where a total removal of fine particulate matter is required. In particular it is also hoped that it may be more efficient in the treatment of fines (Somasundaran, 1986).

The design of a flotation column requires both experimental data about the physical system to be treated in it and a sound theoretical model describing the flow, mixing and mass transfer phenomena occurring in the column. Considerable work has already been done related to the chemistry and operation of flotation columns (see for example, Matis, 1980, 1982, Dobby and Finch, 1985, 1986; Stalidis et al., 1988).

The present work investigates the mixing in a flotation column through the determination of the residence time distribution (RTD) of the liquid phase, since it is known that the efficiency of the flotation process is largely a function of the hydrodynamic characteristics of the system. This is most important for the flotation process, since the state of mixing affects the overall process.

The interpretation of the experimental results is done through a model describing the mixing process in the column; the simplest models for this are either the plug flow model or the completely stirred tank reactors (CSTR) in the series model.

In the first case, mixing is regarded as a deviation from ideal plug flow and a single parameter, the axial dispersion coefficient $D_a$, characterises the degree of mixing in the column. The model is described by an unsteady-state material balance equation written for the tracer:

$$\frac{\partial C}{\partial t} + \frac{\mu_L}{1 - \varepsilon} \frac{\partial C}{\partial x} = D_a \frac{\partial^2 C}{\partial x^2}$$

where $C$ is the concentration of some liquid species, $\mu_L$ is the superficial liquid velocity, $t$ is the time, $x$ is the distance along the column and $\varepsilon$ is the gas hold-up, due to the gas flow in the column (see also the Notation). The value of $D_a$ is determined by comparison of the experimentally obtained RTD curve with the model prediction. Numerous studies have been conducted on various parameters such as the viscosity, the surface tension, the presence of electrolytes, the geometrical features of the column, etc. and excellent reviews exist: see for example Joshi (1980), Shah et al. (1982) and Pandit and Joshi (1983). The model has even been extended so as to obtain both the axial and radial dispersion coefficients, $D_a$ and $D_r$ (see Deckwer and Schumpe, 1987).

On the other hand, the column may be regarded as either one or a series of CSTRs, i.e. well-mixed zones. In this case, the RTD curve is given by the following well known equation:
\[ E = \frac{N \cdot (N \cdot \theta)^{N-1}}{(N-1)!} \cdot e^{-N\theta} \]  

(2)

where \( E \) is the dimensionless RTD, \( N \) is the number of well-mixed zones and \( \theta \) the dimensionless time: \( \theta = t/\tau \) for a mean residence time \( \tau = V/Q_L \), where \( V \) is the volume of the liquid in the column. Since this model allows only for integer values of \( N \), it has been generalised by Buffham and Gibilaro (1968) so that \( N \) can take real values as well:

\[ E = \frac{N \cdot (N \cdot \theta)^{N-1}}{\Gamma(N)} \cdot e^{-N\theta} \]  

(3)

where \( \Gamma(N) \) is the gamma function.

**EXPERIMENTAL DETAILS**

The experimental apparatus presented in Fig. 1 consisted of: a column and its flow accessories, a conductivity cell and corresponding meter, and a recorder.

The column was acrylic-made, approximately 1.00 m high and with an internal diameter of 8.0 cm. It was operated in a countercurrent fashion. The gas-free liquid height was kept constant at 86 cm, corresponding to a liquid volume of 4395 cm\(^3\).

Tap water was fed at the top of the column, at a liquid depth of 4 cm and withdrawn from the bottom of the column, assisted by a peristaltic pump. In
order to assure an even and constant-temperature liquid, this was fed first through a thermostatic bath and then into an overhead constant-level holding tank.

The gas was sparged at the bottom of the column through a sintered-glass disk (Schott, G4), with pores of 10–16 μm (mean diameter). Both gas and liquid flowrates were measured with calibrated flowmeters.

A flow-through conductivity-metering cell, with a constant $k$ equal to 1.05 cm$^{-1}$ was connected to the liquid exit. The connecting tube was kept as short as possible, so as to avoid any effects of the tube volume on the RTD measurement. The cell was connected to a conductivity meter (Metrohm) and the signal from it was recorded on a strip-chart recorder (Hewlett Packard).

A carefully measured amount of solution of 4 M KCl was injected into the liquid stream entering the flotation column, about 1 cm away from the entering point and the recorder was simultaneously started. The time required for the injection was very short, as compared with the mean residence time, so that the injected tracer could be considered as a perfect pulse. The conductivity of the outgoing liquid was recorded for a time period equivalent to approximately three mean residence times and from the obtained trace of conductivity versus time, the residence time distribution was computed.

RESULTS AND DISCUSSION

Experiments were performed in the flotation column for various values of the gas and liquid flowrates (see Table 1). The values of the liquid flowrate were chosen so as to obtain a mean residence time, $\tau$, ranging from a few minutes to the 15–20 min typical of the flotation cells. The gas flowrates, on the other hand, were chosen so as to obtain fine (and possibly non-coalescing) bubbles, as usually existing in flotation vessels.

The RTD curve obtained for the lowest liquid flowrate $Q_L = 200$ cm$^3$ min$^{-1}$,

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
$Q_L$ (cm$^3$ min$^{-1}$) & $Q_G$ (cm$^3$ min$^{-1}$) & $u_L$*1 (cm min$^{-1}$) & $u_G$*2 (cm min$^{-1}$) & $\tau$*3 (min) \\
\hline
200 & 0 & 4.0 & 2.0 & 22 \\
400 & 100 & 8.0 & 6.0 & 11 \\
600 & 300 & 11.9 & 9.9 & 7.3 \\
850 & 500 & 16.9 & 16.4 & 5.2 \\
\end{tabular}
\end{table}

*1 $u_L = Q_L/S$, where $S$: cross-sectional area of column = 50.3 cm$^2$.

*2 $u_G = Q_G/S$.

*3 $\tau = V/Q_L$, $V = 4395$ cm$^3$. 
in the absence of gas, showed that the tracer moved through the flotation column in a more or less plug flow fashion marked by some spreading of the tracer pulse, as shown by curve a in Fig. 2 (where each curve is the average of three replications).

When the gas was introduced, however, the mixing effect was immediately obvious: the turbulence caused by the interaction between the two countercurrent flows mixed the contents of the column and its behaviour soon approached that of a well-mixed vessel.

The increase of gas flowrate affected the mixing process up to a certain limit; for example, at a liquid flowrate of 200 cm$^3$ min$^{-1}$ the effect of the gas flowrate was noticeable only for $Q_G = 100$ and 300 cm$^3$ min$^{-1}$; at the latter, the mixing was complete and the introduction of gas at a higher rate had no additional effect.

The same pattern of behaviour was also displayed at the higher liquid flowrates: Figs. 3, 4 and 5 show the RTD for $Q_L = 400$, 600 and 850 cm$^3$ min$^{-1}$, respectively.

From these figures it is interesting to note that the behaviour of the column, as seen by its RTD, depended also upon the relative magnitude of the liquid and gas flowrates. For instance, at a liquid flowrate of 400 cm$^3$ min$^{-1}$, the "maximum mixing" observed for $Q_G = 300$ cm$^3$ min$^{-1}$ started deviating from the 1-CSTR RTD curve (Fig. 3), whereas this deviation was most obvious for the highest liquid flowrate of 850 cm$^3$ min$^{-1}$ (Fig. 5).

Thus, it is obvious that the simple, one-parameter models could not account for the complex flow features usually encountered in a bubble column, due
Fig. 3. Experimental RTD for $Q_L = 400 \text{ cm}^3 \text{ min}^{-1}$. $Q_G$: $a = 0 \text{ cm}^3 \text{ min}^{-1}$; $b = 100 \text{ cm}^3 \text{ min}^{-1}$; $c = 300 \text{ cm}^3 \text{ min}^{-1}$; $\cdots \cdots \cdot = \text{1 CSTR}$.

Fig. 4. Experimental RTD for $Q_L = 600 \text{ cm}^3 \text{ min}^{-1}$. $Q_G$: $a = 0 \text{ cm}^3 \text{ min}^{-1}$; $b = 100 \text{ cm}^3 \text{ min}^{-1}$; $c = 300 \text{ cm}^3 \text{ min}^{-1}$; $\cdots \cdots \cdot = \text{1 CSTR}$.

mainly to the interaction of the gas and liquid phases as demonstrated by Figs. 2–5. The introduction of the gas at the bottom of the column establishes a circulation pattern in it, with the liquid entrained upwards by the gas in the central core of the column and flowing downwards near the column walls (Freedman and Davidson, 1969; Molerus and Kurtin, 1986). This flow pattern results in a radial non-uniformity of liquid velocity (B.P. Pablov, 1965, and H. Yoshitome, 1967, cited in Ohki and Inoue, 1970; Hills, 1974) and gas-holdup
Fig. 5. Experimental RTD for $Q_L = 850 \text{ cm}^3\text{min}^{-1}$. $Q_G$: $a = 0 \text{ cm}^3\text{min}^{-1}$; $b = 100 \text{ cm}^3\text{min}^{-1}$; $c = 300 \text{ cm}^3\text{min}^{-1}$; $\cdots = 1 \text{ CSTR}$.

(Hills, 1974; Steinemann and Buchholz, 1984; Guy et al., 1986), which depends upon the geometrical factors, the relative flowrates of gas and liquid etc. In the case of flotation columns, the capture and removal of solid particles or particulate matter by the rising bubbles imposes a gradient of particle concentration along the column, which also needs accounting for.

Therefore, it seemed more realistic to visualise the flotation column rather as a network of connected well-mixed zones, with provision for flow of the phases involved from one zone to all adjacent ones. This approach has already been followed for agitated vessels (Mann et al., 1981) and bubble columns (Todt et al., 1977; Jiřičný et al., 1979; Pandit and Joshi, 1983; Deckwer and Schumpe, 1987).

In this context, the simplest possible column configuration comprises a series of well-mixed zones (Fig. 6). Flow is allowed from the $i$th zone to both the $(i+1)$th and the $(i-1)$th zones. The material balance for the tracer in the $i$th zone is:

$$V_i \cdot \frac{dC_i}{dt} = Q_{i-1,i} \cdot C_{i-1} + Q_{i+1,i} \cdot C_{i+1} - (Q_{i,i+1} + Q_{i,i-1}) \cdot C_i \quad (4)$$

where the "$i,i+1$" denotes flow from the $i$th to the $(i+1)$th zone and $V_i$ is the zone volume. There are essentially two parameters in this model: the first one is the number of zones that is used to represent the column flow structure, while the amount of flow flowing backwards, $Q_{i,i-1}$, is the second parameter of the model.

In the case of the bank of flotation cells (Frew, 1984), it has been possible to determine this backflow for each individual zone-to-zone connection. In the
flotation column, however, no such geometrical separation exists and, therefore, a universal value may be considered as characterising the backflow. If $\lambda$ is defined as the ratio of backflow, $Q_{i,i-1}$, to the net liquid flow into the column, $Q_L$, as follows:

$$\lambda = \frac{Q_{i,i-1}}{Q_L}$$

then, the previous material balance equation may be rewritten as:

$$V_i \cdot \frac{dC_i}{dt} = (1 + \lambda) \cdot Q_L \cdot C_{i-1} + \lambda \cdot Q_L \cdot C_{i+1} - (1 + 2\lambda)Q_L \cdot C_i$$

for $i = 2 \ldots N - 1$, whereas for the first and last zone it becomes:

$$V_1 \cdot \frac{dC_1}{dt} = Q_L \cdot C_{in} + \lambda \cdot Q_L \cdot C_2 - (1 + \lambda)Q_L \cdot C_1$$

and:

$$V_N \cdot \frac{dC_N}{dt} = (1 + \lambda) \cdot Q_L \cdot C_{N-1} - (1 + \lambda)Q_L \cdot C_N$$

with initial conditions at time $t = 0$:

$$C_1 (0) = N \cdot C_0 \text{ and } C_i (0) = 0, \quad i = 2 \ldots N$$

where $C_0$ is the initial concentration of the tracer in the whole column and $C_{in}$ is the concentration of the tracer of the incoming liquid flow.

The backflow parameter, thus, is an indication of the degree of mixing in the column: low values of $\lambda$ denote relatively little mixing, whereas large values of
\( \lambda \) means that there is intense mixing of the fluid contents of the various zones, seen here as a backflow of liquid against the main liquid flow.

This model has been used in the past for the interpretation of experimental data from stagewise operations in chemical engineering (Miyauchi and Vermeulen, 1963) and has been compared with the axial dispersion model (Roemer and Durbin, 1967). Recently, however, it has also been considered for modelling the flotation process for the case of communicating banks of agitated flotation cells (Lynch et al., 1981; Frew, 1984).

The eqs. 6–8 may be rewritten in dimensionless and compact form as:

\[
\frac{1}{N} \frac{d}{d\theta} C^* = [A] \cdot C^* + B 
\]

\( C^* \) being an array of dimensionless concentrations:

\[ C_i^* = \frac{C_i}{C_0}, \quad i = 1...N \]  

\( B \) is an array of constants:

\[ B_i = C_{in}, \quad B_i = 0, \quad i = 2...N \]

and \([A]\) is a matrix \( N \times N \) of coefficients:

\[ a_{1,1} = a_{N,N} = -(1+\lambda) \]
\[ a_{i,i+1} = \lambda, \quad i = 1...N-1 \]
\[ a_{i,i-1} = (1+\lambda), \quad i = 2...N \]
\[ a_{i,i} = -(1+2\lambda), \quad i = 2...N-1 \]

it is assumed that all zones have the same volume \( (V_i = V/N) \).

Eq. 9 constitutes a set of ordinary differential equations, which may be solved either numerically or through the Laplace transform for various initial conditions. For example, in the case of a pulse stimulus, at time \( \theta = 0 \):

\[ C_1^*(0) = N \quad \text{and} \quad C_i^*(0) = 0 \quad i = 2...N \]

The curve \( C_N^*(\theta) \), which is obtained from this solution is the theoretical RTD of the column (the case for \( N = 2 \) and \( N = 3 \) is illustrated in the Appendix).

The backflow model has two degrees of freedom: the number of zones \( N \) and the backflow ratio \( \lambda \). One of these parameters, therefore, may be set in advance and the other one is computed by matching the theoretical RTD curve to the experimental data: setting the value of \( N \) and for a given value of \( \lambda \), the sum of squares of the deviation of the theoretical from the experimental value is calculated and the minimum of this sum corresponds to the best fit. For example, in the case of \( Q_L = 200 \text{ cm}^3 \text{ min}^{-1} \) and \( Q_G = 100 \text{ cm}^3 \text{ min}^{-1} \), for \( N = 5 \), the fit was poor for \( \lambda = 1 \) whereas the best fit corresponded to \( \lambda = 10 \), as shown in Fig. 7.

Each experimental condition, therefore, could be fitted by various pairs of \((N, \lambda)\) values, as can be seen in Fig. 8. For all but the lowest liquid flowrate, the
backflow ratio $\lambda$ was lying between 1 and 16 at most, denoting a more or less intense recirculation of the liquid contents of the column. As the number of zones $N$ increased, the backflow ratio increased too, reflecting the need for more backwards flow, in order to represent the mixing of the various zones in the column. It is noted that for the lowest liquid flowrate of 200 cm$^3$ min$^{-1}$, the mixing caused by the introduction of gas was very intense and the behaviour of the column approached soon that of a single CSTR. An attempt to obtain a reasonable fit with this model for $Q_G = 300$ cm$^3$ min$^{-1}$ yielded exceedingly large values of $\lambda$ (> 40), even for a large number of zones.

The backflow parameter seems to reach a limiting value, as the number of zones gets closer to 10; this could be correlated to the observation made by Joshi and Sharma (1979) that in bubble columns the recirculating liquid establishes a pattern of loops, each one having a height approximately equal to the column diameter (later, Zehner (1986) used a similar approach to model the behaviour of a bubble column). Therefore, the number of distinct zones in this case could be given by the ratio of liquid height vs column diameter, which in this case has been kept close to 10.

The intensity of mixing in the column was thus represented by various pairs of $(N, \lambda)$ values, which possibly makes interpretation of the experimental results difficult. It has been proposed, however, to combine these two parameters into a single one, the Peclet number, which is traditionally used for describing the mixing process through the axial dispersion coefficient, $D_a$ (Roemer and Durbin, 1967; Todt et al., 1977):

$$Pe = \frac{2N}{1 + 2\lambda} \left(= \frac{uL}{D_a}\right) \quad (16)$$
where $L$ is the height of the liquid in the column. So, each pair of $(N, \lambda)$ values corresponds to a single value of $Pe$: high values of $Pe$ denote little mixing, i.e. the liquid flow approaches plug flow, whereas low values of $Pe$ correspond to intense mixing of the column contents.

In this way, the effect of the gas and liquid flowrates on the mixing of the flotation column could be clearly demonstrated as shown in Figs. 9 and 10: the increase of $Q_G$ obviously increased the turbulence in the column, therefore causing an increase in the degree of mixing, observed here as a decrease of the $Pe$ number. On the other hand, the increase of $Q_L$ had the opposite effect: the more liquid flowing into the column, the less it was disturbed by the gas flow, hence the degree of mixing decreased, approaching plug flow. This is made obvious in Fig. 11, where effectively the increase of the liquid flowrate is reflected by the increasing remoteness from the 1-CSTR curve.

Figs. 8–10 clearly show that it could be possible to obtain an empirical equa-
Fig. 9. The Pe number as a function of the gas flowrate \( Q_G \). \( Q_G: \bigtriangleup = 400 \text{ cm}^3\text{ min}^{-1}; \nabla = 600 \text{ cm}^3\text{ min}^{-1}; \square = 850 \text{ cm}^3\text{ min}^{-1}. \)

Fig. 10. The Pe number as a function of the liquid flowrate \( Q_L \). \( Q_G: \bigtriangleup = 100 \text{ cm}^3\text{ min}^{-1}; \nabla = 300 \text{ cm}^3\text{ min}^{-1}. \)

Fig. 11. Decrease in mixedness caused by the increase of the liquid flowrate \( Q_L \): \( a = 200 \text{ cm}^3\text{ min}^{-1}; b = 400 \text{ cm}^3\text{ min}^{-1}; c = 600 \text{ cm}^3\text{ min}^{-1}; d = 850 \text{ cm}^3\text{ min}^{-1}. \)

...tion, relating the degree or intensity of mixing (Pe number) or the backflow ratio as a function of: the number of zones and the gas and liquid flowrates. However, no attempt was made, at this stage, to obtain such an equation, since it was thought that more work is required in order to investigate the effect of several other parameters like: (1) the geometrical factors (column diameter and height, position of liquid feed etc.); (2) the size and type of the gas distributor (perforated plate, diffuser or dissolved-air injector); (3) the influence of the various additives (frothers, collectors etc.) added for the enhancement...
of the flotation process; (4) the size, density and concentration of the solids etc., before such a predictive equation could be reliably and safely obtained.

CONCLUSIONS

The degree of mixing in the gas–liquid flotation column, which is of major importance for its performance, is clearly affected by the relative flowrates of the gas and liquid being fed into it, among other parameters. The increase of the gas flowrate has been found to increase the degree of mixing in the column, whereas the increase of the liquid flowrate decreases it.

A relatively simple model was proposed for representing the behaviour of the liquid stream in it. It consists of a series of well-mixed zones, with flow allowed both forward and backwards. The model has two parameters: the number of zones \( N \) and the ratio of backflow to net forward liquid flow \( \lambda \). If an appropriate number of zones is selected, then the behaviour of the column may be predicted for any gas and liquid flowrates, in the range of values for which experiments have been performed.

With this configuration, it should be possible to overlay information about the distribution of values for the solids concentration, the local gas hold-up and bubble size distribution and obtain a full model for predicting the performance of the gas–liquid flotation column.

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NOTATION

\[ a \] : element of matrix \([A]\)
\[ [A] \] : matrix \(N \times N\) of coefficients
\[ B \] : array of constants \([\text{kmol cm}^{-3}]\)
\[ C \] : tracer concentration \([\text{kmol cm}^{-3}]\)
\[ C^* \] : dimensionless concentration
\[ D \] : axial dispersion coefficient \([\text{cm}^2 \text{s}^{-1}]\)
\[ k \] : conductivity cell constant \([\text{cm}^{-1}]\)
\[ L \] : height of liquid in column \([\text{cm}]\)
\[ N \] : number of zones in model
\[ Pe \] : Peclet number \(= u L / D\)
\[ Q \] : net flow into column \([\text{cm}^3 \text{min}^{-1}]\)
\[ t \] : time \([\text{min}]\)
\[ u \] : superficial velocity \([\text{cm min}^{-1}]\)
\[ V \] : column volume \([\text{cm}^3]\)
\[ x \] : distance \([\text{cm}]\)
APPENDIX

The analytical solution of eqs. 9 may easily be obtained when the number of zones-in-series is 2. The unsteady-state mass balance equations for both zones are:

\[ \frac{dC_1}{dt} = -(1 + \lambda) \cdot Q_L \cdot C_1 + \lambda \cdot Q_L \cdot C_2 \]
\[ V_1 \frac{dC_2}{dt} = (1 + \lambda) \cdot Q_L \cdot C_1 - (1 + \lambda) \cdot Q_L \cdot C_2 \]

with initial conditions: \( C_1 (0) = 2 \cdot C_o \) and \( C_2 (0) = 0 \). These may be written in dimensionless form:

\[ \frac{dC_1^*}{d\theta} = -2 \cdot (1 + \lambda) \cdot C_1^* + 2 \cdot \lambda \cdot C_2^* \]
\[ \frac{dC_2^*}{d\theta} = 2 \cdot (1 + \lambda) \cdot C_1^* - 2 \cdot (1 + \lambda) \cdot C_2^* \]

where \( C_1^* (0) = 2 \) and \( C_2^* (0) = 0 \) and they may be transformed now into the Laplace domain as:

\[ s \cdot L(C_1^*) - C_1^* (0) = -2 \cdot (1 + \lambda) \cdot L(C_1^*) + 2 \cdot \lambda \cdot L(C_2^*) \]
\[ s \cdot L(C_2^*) = 2 \cdot (1 + \lambda) \cdot L(C_1^*) - 2 \cdot (1 + \lambda) \cdot L(C_2^*) \]

where \( s \) is the Laplace variable and \( L(C_i^*) \) is the Laplace-transformed \( C_i^* \). From these, \( L(C_2^*) \) may be calculated:

\[ L(C_2^*) = \frac{4 \cdot (1 + \lambda)}{[s + 2 \cdot (1 + \lambda)]^2 - 4 \cdot \lambda \cdot (1 + \lambda)} \]

which is easily inversed into the time domain to yield the theoretical RTD:

\[ C_2^* = \frac{4 \cdot (1 + \lambda)}{[4 \cdot \lambda \cdot (1 + \lambda)]^{1/2}} \cdot \exp \left\{ -2 \cdot (1 + \lambda) \cdot \theta \right\} \]
\[ \times \sinh \left\{ [4 \cdot \lambda \cdot (1 + \lambda)]^{1/2} \cdot \theta \right\} \]

which is a function of dimensionless time \( \theta \) and the backflow ratio \( \lambda \).

In the case of 3 zones-in-series, the analytical solution is still obtainable, after some considerable calculus. The mass balance equations are, in this case:
\[ V_1 \frac{dC_1}{dt} = -(1 + \lambda) Q_L C_1 + \lambda Q_L C_2 \]

\[ V_2 \frac{dC_2}{dt} = (1 + \lambda) Q_L C_1 - (1 + 2\lambda) Q_L C_2 + \lambda Q_L C_3 \]

\[ V_3 \frac{dC_3}{dt} = (1 + \lambda) Q_L C_2 - (1 + \lambda) Q_L C_3 \]

with initial conditions: \( C_1 (0) = 3 C_0 \) and \( C_2 (0) = C_3 (0) = 0 \). These may be non-dimensionalised and written in the Laplace domain:

\[ s L(C_1^*) - C_1^*(0) = -3(1 + \lambda) L(C_1^*) + 3\lambda L(C_2^*) \]

\[ s L(C_2^*) = 3(1 + \lambda) L(C_1^*) - 3(1 + 2\lambda) L(C_2^*) + 3\lambda L(C_3^*) \]

\[ s L(C_3^*) = 3(1 + \lambda) L(C_2^*) - 3(1 + \lambda) L(C_3^*) \]

and \( L(C_3^*) \) is readily obtained:

\[ L(C_3^*) = \frac{27(1 + \lambda)^2}{D} \]

where:

\[ D = (s + 3(1 + \lambda))(s + 3(1 + \lambda))(s + 3(1 + 2\lambda)) - 18\lambda(1 + \lambda) \]

The inversion of this function is still feasible, although lengthy: the denominator is put in the form:

\[ D = (s - a)(s - b)(s - c) \]

where:

\[ a = -3(1 + \lambda), \]

\[ b = 3 + 9\lambda/2 + (9\lambda(8 + 9\lambda))^{1/2}/2 \]

\[ c = 3 + 9\lambda/2 - (9\lambda(8 + 9\lambda))^{1/2}/2 \]

from which \( C_3^* \) is obtained as:

\[ C_3^* = \frac{(b - c) \exp(a \theta) + (c - a) \exp(b \theta) + (a - b) \exp(c \theta)}{(a - b)(b - c)(c - a)} \]

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